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Counting Links and Black Hole Entropy in Causal Sets

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Abstract

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— Wikipedia

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1 Introduction

1.1 Causal sets and Discrete Spacetime

There are many approaches to a theory of quantum gravity. **(ref)** gives a discusion of some of the possibilities. One approach is causal sets which is based on the assumption of fundamental spacetime discreteness.

There are a number of results in current theories that suggest that spacetime might be discrete. The infinities of Quantum Field Theories are caused by the lack of a short distance cut-off. Although these are cured by the renormalisation procedure they return in an unmanageable way in simple attempts to quantise general relativity **(ref)**. The finiteness of black hole entropy also gives another clue. Without a cut-off the entanglement entropy of quantum fields appears to be infinite (black hole entropy will be discussed in more detail in chapter 2). Other approaches to quantum gravity such as Loop Quantum Gravity also suggest fundamental discreteness **(ref)**.

The causal set approach combines the notion of spacetime discreteness with the information gained from the causality relations between events in a spacetime. A theorem by Malament **(ref)** building on earlier work by Hawking **(ref)** states that the metric up to a conformal factor of every future and past distinguishing spacetime can be determined from its causal structure. The points of a weakly causal (future and past distinguishing) Lorentzian manifold and the causal relations between them form a partial ordered set or *poset*. A poset is a set A with the order relation $\prec$ on them that obeys:

(i) Transitivity: $\forall x, y, z \in A, \qquad x \prec y \prec z \Rightarrow x \prec z)$.

(ii) Irreflexifity: $\forall x \in A, \qquad x \not\prec x$.

To impose discreteness the following axiom is added:

(iii) Local finiteness: $\forall x, z \in A, \qquad \mathbf{card}\{y \in A \,|\, x \prec y \prec z\} < \infty$.

Where **card** X is the cardinality of the set X. This last condition ensures that there is a finite number of elements causally between any two elements and so ensures that our poset will be discrete.

A set and order relation that obeys all the above axioms will be a locally finite poset and is called a causal set or *causet.* This can be shown pictorially via a Hasse diagram such as figure **(ref)**. The conformal factor of the metric can be determined from a measure of the volume of spacetime and in the causal set approach this has the interpretation of being simply the number of elements in the set. Therefore a causet includes all the information required to describe a spacetime.

1.2 Recovering the Continuum

Although spacetime is fundamentally discrete our perception on larger length scales is of a continuous Lorentzian manifold. Therefore an important question is when can a Lorentzian manifold (M, g) be said to be an approximation to a causal set?

A causal set A whose elements correspond to points in a spacetime (M, g) and whose partial order is that induced on those points by the causal order of the spacetime is said to be an *embedding* of A into (M, g). Not all causal sets can be embedded into all manifolds and a manifold can not be said to be an approximation of all causal sets that embed into it. There is no volume information, there is no fixed discreteness scale and there may not be enough elements to accurately determine the causal information. So to ensure the correct density of points and to provide volume information it is easier to start from the continuum and work back to the discrete causal set. It is hoped that when a fuller knowledge of the dynamics on a causal set are known the causal sets that approximate relevant manifolds will emerge naturally.

Evidence from black hole entropy **(ref)** implies that the scale on which discreteness becomes apparent is of the order of the plank length. $l_p = \sqrt{\kappa\hbar}$ where $\kappa = 8\pi G$. Therefore the fundamental unit of volume would be $V_f = vl_p^4$ where v is a number of order one. Any causal set whose elements correspond to points in a spacetime such that the number of points in a sufficiently large region is proportional to the volume could be a possible discretised version of the spacetime in question. A simple way of doing this would be to set up a lattice of points in spacetime whose intervals are separated by $v^{1/4}l_p$. The problem with this method is that it is coordinate dependant and therefore not Lorentz invariant. Under highly boosted frames large voids where there are no points at all will occur **(ref)**. Although in the original frame and ones related by a sufficiently small boost the causal set would still resemble a Lorentzian manifold, in highly boosted frames if this structure was fundamental there would be no manifold description at all.

Since Lorentz invariance is not seen to be violated a distribution of points in spacetime that is Lorentz invariant is required. Such a distribution is found in the Poisson distribution. The *sprinkling* process is essentially a process where elements of the causet A are placed in the spacetime via a Poisson process with density ρ. Therefore the probability of finding n elements in a given volume V is:

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \tag{1.1}$$

Where ρ is of Plankian order. A Lorentzian manifold (M, g) is said to approximate a causal set A if A could have come from sprinkling (M, g) with sufficiently high probability. If this is the case a is said to be *faithfully embedded* in m.

1.3 Basic Definitions

- $J^+(p)$ is the causal future of the set p and is the set of all elements r such that $p \prec r$.
- $J^-(q)$ is the causal past of the set q and is the set of all elements r such that $r \prec q$.
- An *interval (Alexandrof set)* $J(x, y)$ between two elements x, y such that $x \prec y$ is the intersection $J^+(x) \cap J^-(y)$. It is the set of all elements r such that $x \prec r \prec y$.
- An element p is *maximal* in a causal set C if there are no elements q such that $p \prec q$.
- An element p is *minimal* in a causal set C if there are no elements q such that $q \prec p$
- A *link* is between two elements whose interval is empty.

2 Black Hole Thermodynamics

2.1 Link Counting

In statistical mechanics the entropy is found to be proportional to the number of microstates contained in the system. In the classical limit and for a perfect gas this reduces to counting the number of molecules to leading order. The philosophy behind link counting is that the corresponding horizon "molecules" for black hole entropy are the links that cross the horizon. This makes sense heuristically as a link can be seen as the flow of information from one space time point to another. Therefore counting links across the horizon would hopefully give a measure of the amount of information that can flow into the interior of the black hole.

The hope therefore is that by counting links with suitable conditions a result that is proportional to the area of the horizon will be found. Furthermore the relationship between black hole entropy and horizon area appears to be the same for all horizons, including cosmological. Therefore to be a possible interpretation of black hole entropy our program must give a result that has the same constant of porportionality for all horizons.

For a causal set to approximate a sapcetime on large scales it must be fiathfully embedded. This can be done by random sprinkling on the relevant spacetime giving a distribution of points as given in equation 1.1.

(derivation of basic integral)

For a given horizon the area to which the entropy is proportional to is given by the area of the intersection of the horizon with a hypersurface (Σ).

$$S = k.A(H \cap \Sigma)$$

The hypersurface can be null or spacelike. The links we be the ones that cross both the horizon and the hypersurface. Therefore the links will be between points $x \in J^-(H) \cap J^-(\Sigma)$ and $y \in J^+(H) \cap J^+(\Sigma)$. The number of links between points in these regions is infinite and is shown in the case of a flat space time in chapter 3. This divergence is understood to be due to the infinite number of links that are null related and extended along the horizon and hypersurface. Therefore further conditions are required to suppress this divergence.

In **(ref)** Sorkin and Duo argue that we intuitively want to estimate the amount of information flowing across the horizon at a given time and not the total amount. Therefore to count a given link for more than one hypersurface would be to overcount it. The links that are extended along the horizon will contribute to many hypersurfaces and so would seem to be responsible for the divergence. Therefore the conditions that are required will be ones that only count links that belong to the hypersurface in question and not to ones earlier or later.

One condition that should do this is forcing x to be maximal in $J^-(\Sigma)$. Links corresponding to x being far away from Σ should be suppressed by the extra volume in the exponential. Therefore the contributing links will not cross a earlier hypersurface and would be suppressed for a hypersurface occurring later. But other conditions will exist that suffice and there is no natural reason why the above one should be used. In chapter 3 it will be shown that the maximality condition on x has the same form as a similar minimality condition on y for the case of a collapsing null shell.

These conditions will be further examined in chapter 3. Below I include a calculation carried out by Duo in **(ref)** that shows how the minimality condition on y gives rise to a finite answer for a dimensionally reduced Schwarzchild black hole with null hypersurface.

2.2 2D Schwarzchild

Ideally the calculation would be caried out on a full four dimensional Schwarzchild black hole but due to the complexity of the non radial goedesics it is hard to evaluate the Alexandrov volume of two points let alone the volume corresponding to the extra conditions. Therefore a dimensionally reduced two dimensional Schwarzchild spacetime where the angular coordinates are suppressed is considered.

To simplify the calculation the presence of the collapse is ignored and the hypersurface is chosen to intersect the horizon far from the collapse. The hypersurface is chosen to be null and the portion of spacetime used in the calculation is shown in....

The metric describing this spacetime with suppressed angular coordinates is:

$$ds^2 = -\frac{4a^3}{r}e^{-r/a}dudv$$

where a is the Schwarzchild radius of the black hole $(2.M)$ and u and v are the Kruskal-Szekeres coordinates with r implicitly defined via:

$$uv = (1 - \frac{r}{a})e^{r/a}$$

The volume element is:

$$d^2V = \sqrt{-g}dudv = \frac{2a^3}{r}e^{-r/a}dudv$$

The ingoing null hypersurface is defined by the equation $v = v_0$. The horizon will correspond to $u = 0$. The conditions impossed on the points (x, y) giving rise to our links is:

$$\begin{gathered} x \in J^-(H) \cap J^-(\Sigma) \\ y \in J^+(H) \cap J^+(\Sigma) \\ x \prec y \text{ is a link} \\ x \text{ maximal in } J^-(H) \cap J^-(\Sigma) \\ y \text{ minimal in } J^+(H) \end{gathered}$$

For the null Σ used here the fourth condition is redundant.

The volume of the excluded region corresponding to the above conditions is:

$$V = a^2 + r_{xy}^2 - r_{xx}^2 - r_{yy}^2$$

where:

$$u_i v_j = (1 - \frac{r_{ij}}{a})e^{r_{ij}/a}$$

The expected number of links satisfying these conditions is therefore:

$$< n >= (2a^3)^2 \int_0^{v_0} dv_x \int_{-\infty}^0 du_x \int_{v_0}^\infty dv_y \int_0^{1/v_y} du_y \frac{e^{-r_{xx}/a - r_{yy}/a}}{r_{xx} r_{yy}} e^{-V}$$

Changing integration variables from (u_x, v_x, u_y, v_y) to $(r_{xx}, r_{x0}, r_{xy}, r_{yy})$, and then by the substitutions $x = r_{xy}$, $y = r_{x0}$ and $z = r_{xx}$ reduces the integral to:

$$< n >= 4\, I(a)\, J(a)$$

where

$$I(a) = \int_a^\infty dx \frac{x}{x-a} e^{-x^2} \int_a^x dy \frac{y}{y-a} \int_a^y dz\, e^{z^2}$$

and

$$J(a) = e^{-a^2} \int_0^a dr_{yy}\, e^{r_{yy}^2}$$

The black holes of interest will be ones for which $a \gg 1$. In this regime:

$$I(a) = \frac{\pi^2}{12}\, a + O\Big(\frac{1}{a}\Big)$$

and

$$J(a) = \frac{1}{2a} + O\Big(\frac{1}{a^3}\Big)$$

Therefore:

$$< n > = \frac{\pi^2}{6} + O\Big(\frac{1}{a}\Big)$$

This is finite and therefore looks promising. At first glance the integrals appear to be dominated by points near horizon and so is said to be controlled by the near horizon geometry. In four dimensions this is locally the two dimensions used here multiplied by the euclidean plane. Therefore where as here $< n >$ is proportional to a point in four dimensions it would be expected to be multiplied by the angular coordinates representing the euclidean plane and therefore be proportional to the horizon area. Unfortunately this is not the case as will be argued in section 3.4.

For link counting to be a possible source of horizon entropy it must work for different types of horizons. A horizon whose spacetime is particularly easy to study is the collapsing null shell since it gives rise to a horizon with a flat spacetime inside the shell. What follows in the later chapters will therefore be a study of the effect of different conditions on the link counting procedure in this spacetime.

3 Basic link counting in a flat spacetime

Due to Birkoff's theorem any spherically symmetric solution of the Einstein equations must be stationary and asymptotically flat and therefore its exterior solution must be given by the Schwarzchild metric. If you consider a collapsing null shell the interior will posses a flat Minkowski metric and it will collapse to a Schwarzchild black hole due to the above theorem.

The horizon is defined as the boundary of the causal past of future infinity. Therefore the horizon will form inside the shell at a time before the shell has passed its Scharzchild radius. This is because light within this region will only be able to reach the Schwarzchild radius after the null shell has passed this radius and the black hole has been formed. At this point light can no longer escape and so can not reach future infinity. Therefore this gives rise to a horizon in the flat spacetime within the shell. The metric in normalised (u, v) coordinates will be:

$$ds^2 = -dudv + r^2 d\Omega^2$$

The spacetime diagram is shown in figure 3.1. The null coordinates (u, v) are chosen to be such that the horizon first forms at the origin. The null shell collapses to the point (b, b). The hypersurface (Σ) intersects the horizon at $(0, a)$ and $(a, 0)$. The four dimensional spacetime has been dimensionally reduced so that each point correspond to halves an S^2. Antipodal points on the sphere are shown. This is because some of the calculations are simpler in this scheme than where the full S^2 angular components are suppressed. This should have no effect on the validity of the calculations.

Since we will be considering the dimensionally reduced 2D case the angular components $(r^2 d\Omega^2)$ will be dropped. As with the Schwarzchild case we are interested in macroscopic black holes. Also since ideally most of the links counted should be close to the intersection of H and Σ there should be a negligible contribution from links outside the shell if Σ crosses H at a sufficiently early enough time. Therefore $b \gg a \gg 1$ if the above two conditions hold.

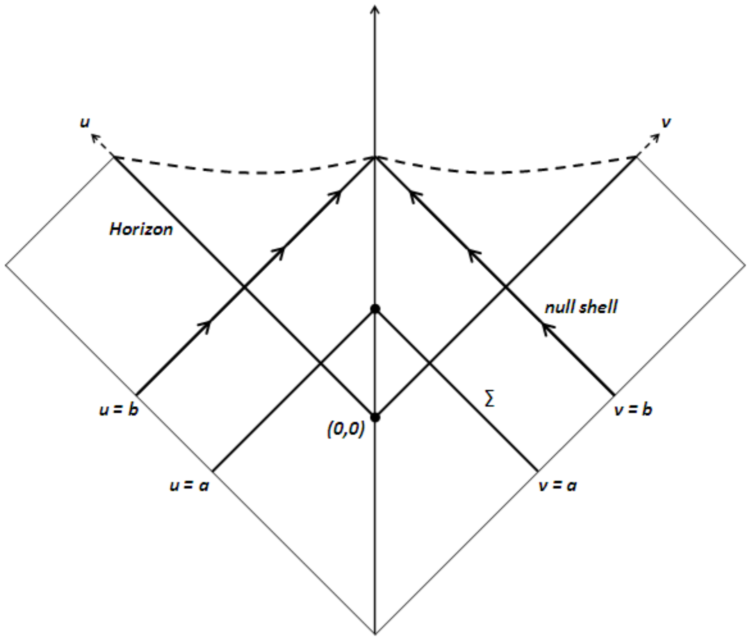


Figure 3.1: Penrose diagram for the collapsing null shell. (u, v) coordinates are chosen so that the horizon forms at the origin

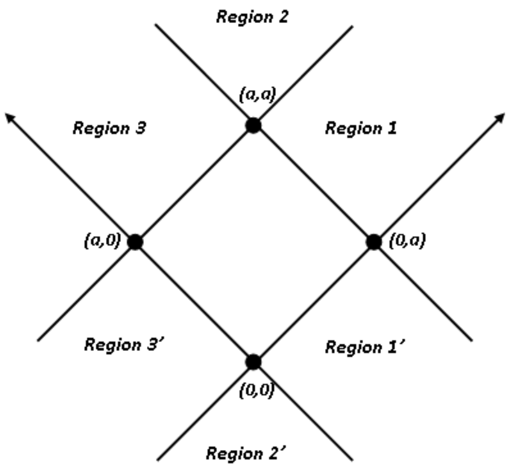


Figure 3.2: Regions of integration

The volumes required to be empty will depend on where the points x and y exist. So I have split the space into several regions as shown in figure 3.2. Requiring only that $x \in J^-(H) \cap J^-(\Sigma)$ and $y \in J^+(H) \cap J^+(\Sigma)$ will force x to exist in regions 1',2' or 3' and y to exist in regions 1,2 or 3. This same classification of regions will be used for all the calculations that follow.

3.1 The Causal Diamond

The expected number of links will again have the form of equation **(ref)**. If no further conditions are imposed except that $x \in J^-(H) \cap J^-(\Sigma)$ and $y \in J^+(H) \cap J^+(\Sigma)$ form a link. The volume required to be empty will be the interval between the two points. Sometimes referred to as the causal diamond.

The integration range for v_y will be taken to infinity. Strictly speaking it should be limited by the shell ($v = b$) but it is assumed that links far away from Σ will make a marginal contribution. Therefore the inclusion of these links should have a limited effect.

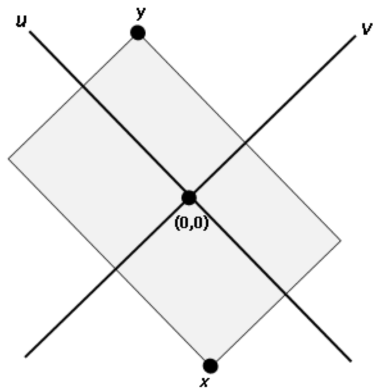


Figure 3.3: The causal diamond

$$< n > = \int_0^a dv_x \int_a^\infty dv_y \int_0^\infty du_y \int_{-\infty}^0 du_x \, e^{u_x(v_y - v_x) - u_y(v_y - v_x)}$$

integrating over u_x and u_y gives:

$$< n > = \int_0^a dv_x \int_a^\infty dv_y \frac{1}{(v_y - v_x)^2}$$

$$< n > = ln(a) - ln(0) = \infty$$

This divergence is due to the large number of links that are close to being null. The believe is that these links will make a contribution to many different hypersurfaces and so in a sense are counted more than once. The attempts to produce a finite answer for link counting are based around the idea of trying to only count links which are associated with a given hypersurface. Therefore conditions must be imposed that will suppress the links that cross many hypersurfaces i.e. the ones that are null or close to null. One way of doing this is by forcing the points x and y to be close to the intersection of Σ and H.

3.2 Min y

In an attempt to force the links to be close to the intersection of Σ and H a proposal was made to add the condition that x is maximal in $J^-(\Sigma)$. This particular choice of a maximality condition on x will be referred to as *min x*. The effect of this extra condition will be that a greater volume of spacetime will be required to be empty of points after the sprinkling process. This volume will appear in the exponential and will have the effect of confining v_x to be close to Σ.

The hope is that since the point x is confined to be close to Σ the links counted using these conditions will be associated to a particular Σ and will not make a contribution to Σ', a hypersurface occurring earlier or later. If Σ' occurs earlier these links will not be counted as v'_x (v_x in relation to this new hypersurface) will now be greater than a' (the new point of intersection between the hypersurface and the horizon) and so the link will not cross this hypersurface. If Σ' occurs later these links should again not be counted as v'_x will be too far away from Σ' and will thus be suppressed.

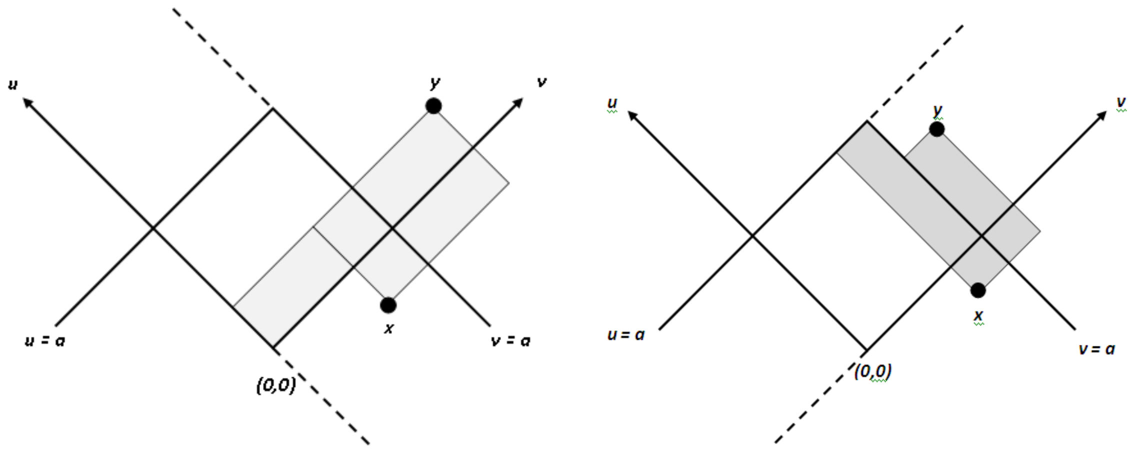
Since the hypersurface used here is null there is a symmetry between this condition and the condition that y is minimal in $J^+(H)$. This condition will be referred to as *min y*. The max x condition is related to the min y condition by changing variables as: $u \leftrightarrow -v$, $x \leftrightarrow y$ and moving the origin from $(0,0)$ to (a,a). The condition of min y has the effect of confining u_y to be close to the horizon and prevents contributions from deep inside the black hole. The volumes and integrals for the min y condition are often simpler and therefore from now on the min y condition will be used. Therefore the conditions used will be:

$$\begin{gathered} x \in J^-(H) \cap J^-(\Sigma) \\ y \in J^+(H) \cap J^+(\Sigma) \\ x \prec y \text{ is a link} \\ y \text{ minimal in } J^+(H) \end{gathered}$$

The Volume(V) required to be empty in the case of min y depends on whether x is in region 1' or region 2'.

if x is in region 1': $\quad V = u_y.v_y - u_x(v_y - v_x)$
if x is in region 2': $\quad V = u_y(v_y - v_x) - u_x(v_y - v_x)$

x in Region 1' - y in Regions 1 or 2



(a) min y in region 1'-12 (b) max x in region 1',2'-1

Figure 3.4: comparison of the min y and max x conditions

This region includes the intersection of Σ and H and so it would be expected that it would be responsible for the majority if not the whole contribution to the total average number of links counted given our constraints. The average number of links coming from this region will be:

$$< n >= \int_0^a dv_x \int_a^\infty dv_y \int_0^\infty du_y \int_{-\infty}^0 du_x \, e^{u_x(v_y - v_x) - u_y.v_y}$$

$$< n >= \int_0^a dv_x \int_a^\infty dv_y \, \frac{1}{v_y(v_y - v_x)}$$

$$< n >= \int_a^\infty dv_y \, \frac{1}{v_y} ln(\frac{v_y}{v_y - a})$$

Using the series expansion for $ln(1-x)$ for small x gives:

$$< n >= \int_a^\infty dv_y \sum_{n=1}^\infty \frac{1}{n} \frac{a^n}{v_y^{(n+1)}}$$

$$< n >= \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

This is finite and agrees with the dimensionally reduced Schwarzchild result. This is promising. Due to the fact that Black hole entropy appears to be of the same form regardless of the type or characteristics of the black hole then the results gained by link counting should be the same as well.

To see where these links are coming from it is interesting to include a cut off. If v_y is restricted to be less than $b.a$, where b is a positive factor greater than 1 the integral becomes:

$$< n >= \int_a^{b.a} dv_y \frac{1}{v_y} ln(\frac{v_y}{v_y - a})$$

Again using the series expansion for $ln(1-x)$ gives:

$$< n >= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{b^n}$$

The series in the second term represents the number of links between the cutoff and infinity. It converges to:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{b^n} = Li_2(1/b)$$

Similarly if a cut off is introduced on v_x as v_x been restricted to be greater than a/b then the integral becomes:

$$< n >= \int_{a/b}^{a} dv_x \int_a^{\infty} dv_y \frac{1}{v_y(v_y - v_x)}$$

$$< n >= \int_a^{\infty} dv_y \frac{1}{v_y} ln(\frac{v_y - a/b}{v_y}) + ln(\frac{v_y}{v_y - a})$$

$$< n >= \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\Big(\frac{a/b}{v_y}\Big)^n - \Big(\frac{a}{v_y}\Big)^n \Big]_a^{\infty}$$

$$< n >= \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{b^n}$$

This is the same as the result for a cutoff on v_y. A plot of the dependence of $< n >$ on this cutoff is shown in figure 3.5. The $< n >$ has been rescaled so that when $< \tilde{n} >= 1$ $< n >= \pi^2/6$.

If both cut offs are applied at the same time the number of links counted is:

$$< n >= \int_{a/b}^{a} dv_x \int_a^{b.a} dv_y \frac{1}{v_y(v_y - v_x)}$$

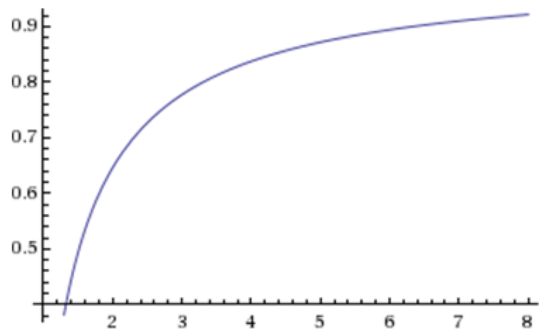


Figure 3.5: rescaled $< n >$ with only one cutoff.

$$< n >= \sum_{n=1}^{\infty} \frac{1}{n^2} \Big[\Big(\frac{a/b}{v_y} \Big)^n - \Big(\frac{a}{v_y} \Big)^n \Big]_a^{a.b}$$

$$< n >= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} (\frac{1}{b^{2n}} - \frac{2}{b^n})$$

$$< n >= \frac{\pi^2}{6} + Li_2(1/b^2) - 2.Li_2(1/b)$$

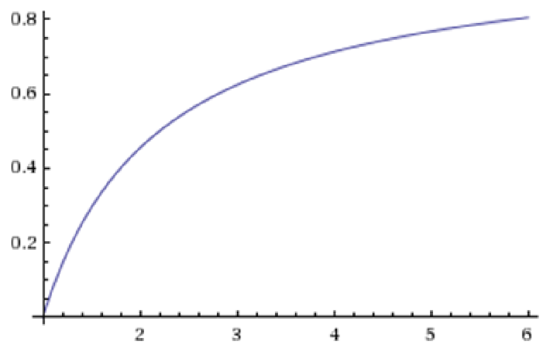


Figure 3.6: rescaled $< n >$ with both cutoffs

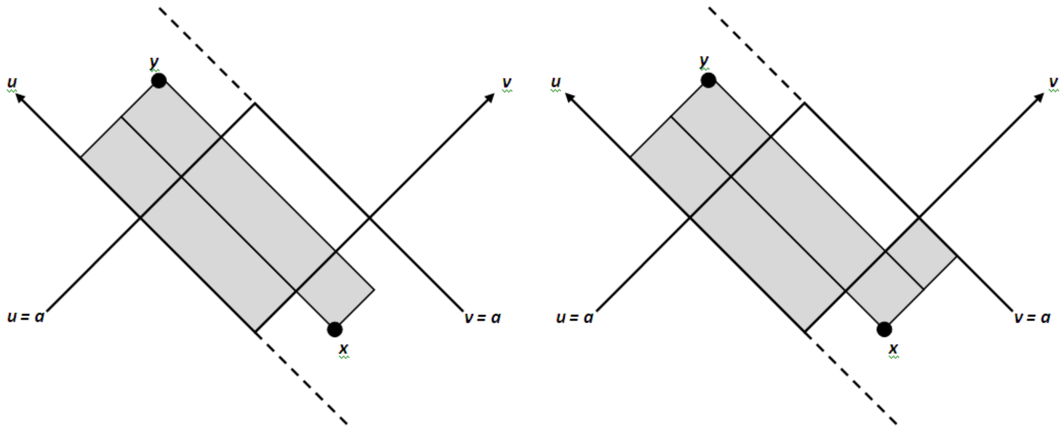
A plot of the dependence of $< \tilde{n} >$ on both cutoffs applied at the same time is shown in figure 3.6.

The plots of $< n >$ show that the average number of links gradually approaches $\pi^2/6$. If a is large this shows that links far away from Σ are contributing to our result. The divergence suppressed by imposing the min y condition is of links extending deep into the black hole interior and so maybe it should not be surprising that the resulting links extend far along the horizon.

The argument for the divergence found in the causal diamond was that it was due to over counting links that cross many hypersurfaces. The links calculated using the min y condition as shown above appear to still cross many horizons and so there is little reason to believe that these links are particularly associated with any one horizon. The reason why the divergence is still cured is that the links close to the intersection are more numerous and so contribute with a greater weight. If the calculation above was followed through with a max x condition as initial proposed then the contributing links would extend far along Σ instead. These links would then extend far into the interior of the Black Hole.

The contribution from x and y in other regions was assumed to be negligible compared to this region. But given that v_x is not necessarily confined to be close to Σ this may not be the case.

x in region 1' - y in region 3



(a) Without a maximality condition on x (b) With x max in $J^-(\Sigma) \cap J^-(H)$

Figure 3.7: Volumes for integration in Region 1'-3

The figure showing the volume corresponding to x and y in these regions is shown in figure 3.7(a). The integral for this case is:

$$< n > = \int_0^a dv_y \int_0^{v_y} dv_x \int_a^\infty du_y \int_{-\infty}^0 du_x \, e^{u_x(v_y - v_x) + u_y . v_x}$$

$$< n > = \int_0^a \frac{dv_y}{v_y} \int_a^{v_y} dv_x \frac{1}{(v_y - v_x)} e^{a.v_y}$$

$$< n > = \int_0^a \frac{dv_y}{v_y} e^{a.v_y} \Big(ln(v_y - a) - ln(0) \Big) = \infty$$

The problem here is that when v_x tends to v_y and v_y tends to 0 the min y condition becomes less and less effective. In the limit of $v_y = 0$ the condition has no effect and there is nothing stoping u_y running off to infinity and producing a divergent number of counted links. These links are again far away from the point of interest and so intuitively should make only a small contribution if any at all to the final answer.

This is why a maximality condition is required on x as well. Applying the max x condition at the same time as the min y condition should have the desired effect of suppressing this divergence. This will be examined in section 3.3. But first I will examine the effects of a less strict condition of making x minimal in $J^-(\Sigma) \cap J^-(H)$. The full list of conditions is now:

$$\begin{gathered} x \in J^-(H) \cap J^-(\Sigma) \\ y \in J^+(H) \cap J^+(\Sigma) \\ x \prec y \text{ is a link} \\ y \text{ minimal in } J^+(H) \\ x \text{ maximal in } J^-(H) \cap J^-(\Sigma) \end{gathered}$$

These are the same conditions used by Duo and Sorkin in **(ref)**. The volume required to be empty to satisfy these conditions is:

$$V = u_x(v_x - a) + u_y.v_y$$

Therefore the integral required to evaluate the average number of links is now:.

$$< n >= \int_0^a dv_y \int_0^{v_y} dv_x \int_a^{\infty} du_y \int_{-\infty}^{0} du_x \, e^{u_x(a-v_x)-u_y.v_y}$$

$$< n >= \int_0^a dv_y \int_0^{v_y} dv_x \, \frac{1}{v_y} \frac{1}{(a-v_x)} e^{-a.u_y}$$

$$< n >= \int_0^a dv_y \, \frac{\big(ln(a) - ln(a-v_y)\big) e^{-a.u_y}}{v_y}$$

$$< n >= \left[ln\Big(\frac{a-v_y}{a}\Big) \frac{e^{-a.v_y}}{a} + Ei(-a(a-v_y)) \frac{e^{-a^2}}{a} \right]_0^a$$

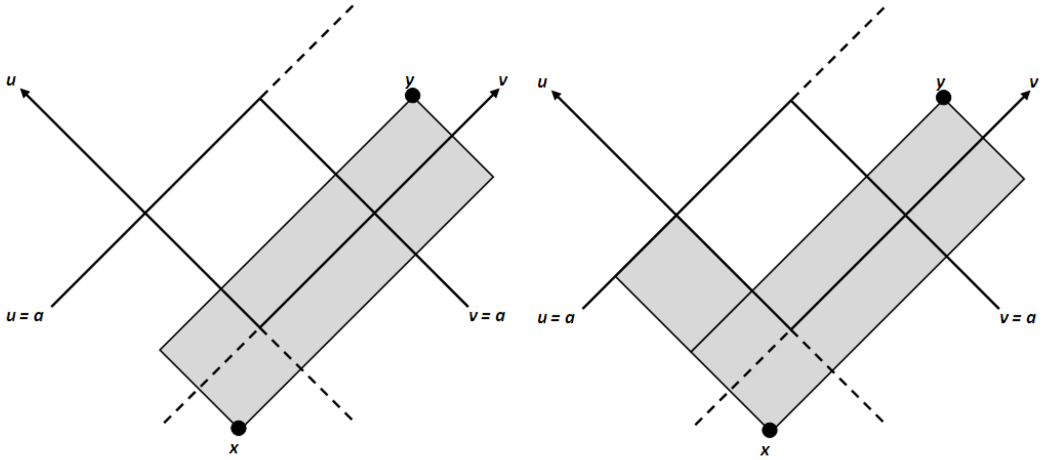
$$< n >= (Ei(a^2) + 2log(a) + \gamma) \frac{e^{-a^2}}{a}$$

This is suppressed for large a and so the divergence has been successfully dealt with.

This extra condition on x is redundant in the earlier cases as the volume it adds was already contained within the existing conditions. Therefore the previous calculations are unchanged.

x in region 2' - y in region 1

First I will present the calcultion for this case without any maximality condition on x.



(a) Without a maximality condition on x (b) With x max in $J^-(\Sigma) \cap J^-(H)$

Figure 3.8: Volumes for integration in Region 2'-1

For x and y in these regions the min y condition is redundant and the volume is the same as that used in the causal diamond calculation.

$$<n>=\int_a^\infty dv_y \int_{-\infty}^0 dv_x \int_0^\infty du_y \int_{-\infty}^0 du_x \, e^{u_x(v_y-v_x)-u_y(v_y-v_x)}$$

integrating over u_x, u_y and v_x gives:

$$<n>=\int_a^\infty \frac{1}{v_y} = \infty$$

This divergence is similar to the one found for region 1'-3. As u_x tends to 0 there is nothing stopping v_y and v_x running off to infinity and -infinity respectively. As before the same extra condition of x being minimal in $J^-(\Sigma) \cap J^-(H)$ is enough to suppress this divergence. The integral now becomes:

$$<n>=\int_a^\infty dv_y \int_{-\infty}^0 dv_x \int_0^\infty du_y \int_{-\infty}^0 du_x \, e^{-v_y(u_y-u_x)+v_x(a-u_x)}$$

$$<n>=\int_0^\infty du_y \int_{-\infty}^0 du_x \frac{1}{(a-u_x)(u_y-u_x)} e^{-a(u_y-u_x)}$$

$$< n >= \int_{-\infty}^{0} du_x \int_{-u_x}^{\infty} dy \frac{e^{-a.y}}{y}$$

$$< n >= \int_{-\infty}^{0} du_x \frac{1}{(a-u_x)} Ei(a.u_x) \leq \frac{1}{a}$$

x in region 2' - y in region 2

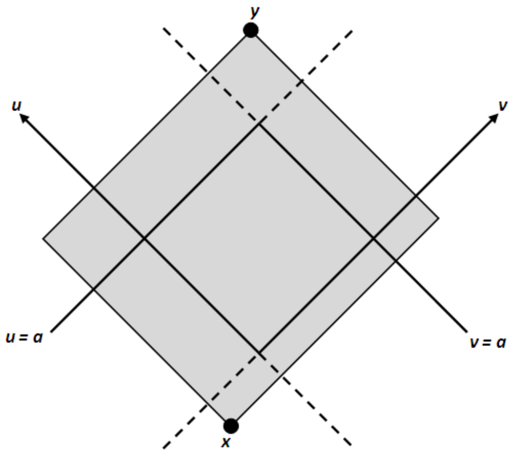


Figure 3.9: volume for integration in Region 2'-2

This diagram should be suppressed. This can be seen due to the fact that the minimal value the volume in the exponential could take is a^2. Therefore this is expected to be suppressed as a becomes large.

$$< n >= \int_{a}^{\infty} dv_y \int_{-\infty}^{0} dv_x \int_{a}^{\infty} du_y \int_{-\infty}^{0} du_x \, e^{u_x(v_y-v_x)-u_y(v_y-v_x)}$$

$$< n >= \int_{a}^{\infty} dv_y \int_{-\infty}^{0} dv_x \, \frac{e^{-a(v_y-v_x)}}{(v_y-v_x)^2}$$

$$< n >= \int_{a}^{\infty} dv_y \, (\frac{e^{-a.v_y}}{v_y} - a.Ei(a.v_y))$$

$$< n >= (1+a^2)Ei(a^2) - e^{-a^2}$$

This is suppressed for large a as expected.

3.3 Min y and Max x

To get rid of the divergences seen in the original causal diamond calculation there is also the possibility of imposing the min y and max x simultaneously. The use of both these conditions will suppress all the divergences shown so far and has the added aesthetic benefit of being (at least in the case of the null hypersurface) symmetric in its approach to both the points x and y. The list of conditions is now:

$$
\begin{gathered}
x \in J^-(H) \cap J^-(\Sigma) \\
y \in J^+(H) \cap J^+(\Sigma) \\
x \prec y \text{ is a link} \\
y \text{ minimal in } J^+(H) \\
x \text{ maximal in } J^-(\Sigma)
\end{gathered}
$$

The volumes corresponding to these conditions is now are different for each of the distinct regions. (Here again cases where x is in region 3' are related to ones where x is in region 1'. Also again region 2'-3 gives the same result as 2'-1).

Region 1'-1

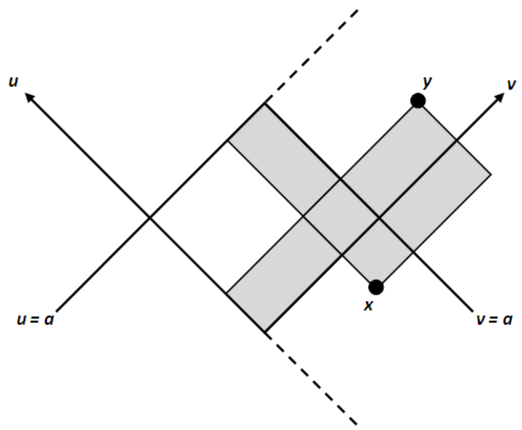


Figure 3.10: volume for integration in Region 1'-1

This region is the most important as the principle behind these extra conditions is to attempt to force the links to be localised around the intersection of H and Σ. This localised region is contained here therefore this contribution should provide the majority of the links counted.

The volume required to be empty is:

$$v = u_y(v_y + v_x) - u_x(v_y - v_x) - a(u_y + v_x) + a^2$$

$$< n >= \int_0^a dv_x \int_a^\infty dv_y \int_0^a du_y \int_{-\infty}^0 du_x \, e^{-u_y(v_y+v_x)+u_x(v_y-v_x)+a(u_y+v_x)-a^2}$$

$$< n >= \int_0^a dv_x \int_a^\infty dv_y \, \frac{1}{(v_y - v_x)} \frac{1}{(v_y - a + v_x)} (1 - e^{-a(v_y+v_x-a)}) e^{-a(a-v_x)}$$

Separating the integral into 2 parts:

$$I_a = \int_0^a dv_x \int_a^\infty dv_y \, \frac{1}{(v_y - v_x)} \frac{1}{(v_y - a + v_x)} e^{-a(a-v_x)}$$

$$I_b = -\int_0^a dv_x \int_a^\infty dv_y \, \frac{1}{(v_y - v_x)} \frac{1}{(v_y - a + v_x)} e^{-a.v_y}$$

Looking at I_a first.

$$\frac{1}{(v_y - v_x)} \frac{1}{(v_y - a + v_x)} = \Big(\frac{1}{(v_y - v_x)} - \frac{1}{(v_y - a + v_x)} \Big) \frac{1}{(2v_x - a)}$$

Using this and to integrate over v_y gives:

$$I_a = \int_0^a dv_x \, \frac{e^{-a(a-v_x)}}{(2v_x - a)} ln\Big(\frac{v_x}{a - v_x} \Big)$$

This is difficult to evaluate. But limits can be placed on the integral.
Now back to I_b.
Similar to the technique used for I_a, integrating over v_x using:

$$\frac{1}{(v_y - v_x)} \frac{1}{(v_y - a + v_x)} = \Big(\frac{1}{(v_y - v_x)} + \frac{1}{(v_y - a + v_x)} \Big) \frac{1}{(2v_y - a)}$$

gives:

$$I_b = \int_a^\infty dv_y \Big(ln\Big(\frac{v_y}{v_y - a} \Big) - ln\Big(\frac{v_y - a}{v_y} \Big) \Big) \frac{e^{-a.v_y}}{(2v_y - a)}$$

$$I_b = \int_a^\infty dv_y \frac{2}{(2v_y - a)} ln\Big(\frac{v_y}{v_y - a} \Big) e^{-a.v_y}$$

Using series expansion in a/v_y gives:

$$I_b = \int_a^\infty dv_y \, 2 \sum_{k=0}^{\infty} \frac{a^k}{(2v_y)^{k+1}} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{a}{v_y} \right)^n e^{-a.v_y}$$

$$I_b = \int_a^\infty dv_y \left[\frac{a}{v_y^2} + \frac{a^2}{v_y^3}(1+\frac{1}{2}) + \frac{a^3}{v_y^4}(1+\frac{1}{2}+\frac{1}{3}) \dots \frac{a^p}{v_y^{p+1}} \sum_{q=1}^{p} \frac{1}{q} \left(\frac{1}{2} \right)^{p-q} \dots \right] e^{-a.v_y}$$

Therefore:

$$I_b \leq e^{-a^2} \int_a^\infty dv_y \sum_{p=1}^{\infty} \frac{a^p}{v_y^{p+1}} \sum_{q=1}^{p} \frac{1}{q} \left(\frac{1}{2} \right)^{p-q}$$

$$I_b \leq -e^{-a^2} \left[\sum_{p=1}^{\infty} \frac{a^p}{v_y^p} \sum_{q=1}^{p} \frac{1}{p.q} \left(\frac{1}{2} \right)^{p-q} \right]_a^\infty$$

$$I_b \leq -e^{-a^2} \sum_{p=1}^{\infty} \sum_{q=1}^{p} \frac{1}{p.q} \left(\frac{1}{2} \right)^{p-q}$$

This will be suppressed as long as the sum converges. To show that it converges it is enough to consider only two cases. The first when $q = 1$ and the second when $q = p$. The series with q valued between these should converge to values between these two extreme cases.

First when $q = 1$:

$$\sum_{p=1}^{\infty} \frac{1}{p} 2^{-p} = log(2) \approx 0.693147$$

Second when $q = p$:

$$\sum_{p=1}^{\infty} \frac{1}{p^2} = \frac{\pi^2}{6}$$

Therefore this contribution is exponentially suppressed in a. It appears that by these more stringent conditions exclude not just the divergent links but almost all the links that we wish to count.

This appears to rule out this case as a useful means of calculating links. This contribution should be the most important as discussed earlier. There is the possibility that the suppressed contribution of this region may counteract the divergence found in the 4-D case. But this appears unlikely since the nature of the suppression here is different to the nature of the divergence in 4-D.

The contributions from the other regions are expected to be suppressed. They are and a proof of this is given below.

Region 1'-2

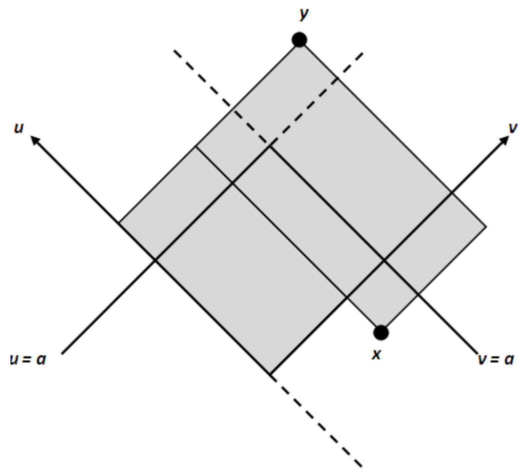


Figure 3.11: volume for integration in Region 1'-2

The volume required for this region is:

$$v = v_y(u_y - u_x) + u_x.v_x$$

In this region the minimality condition on x is redundant and the volume reduces to that of the min y case.

$$< n >= \int_0^a dv_x \int_a^\infty dv_y \int_a^\infty du_y \int_{-\infty}^0 du_x\, e^{-u_y.v_y + u_x(v_y - v_x)}$$

$$< n >= \int_0^a dv_x \int_a^\infty dv_y \frac{1}{v_y(v_y - v_x)} e^{-a.v_y}$$

$$< n >= -\int_a^\infty dv_y\, ln\Big(\frac{v_y - a}{v_y}\Big)\frac{1}{v_y} e^{-a.v_y}$$

Using the series expansion for $ln(1 - x)$ for small x:

$$< n >= \int_a^\infty dv_y\, e^{-a.v_y} \frac{1}{v_y} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{a}{v_y}\right)^n$$

Approximating by taking the maximal value of the exponential and then integrating gives:

$$< n > \leq -e^{-a^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\frac{a^n}{v_y^n} \right]_a^{\infty} \leq \frac{\pi^2}{6} e^{-a^2}$$

Region 1'-3

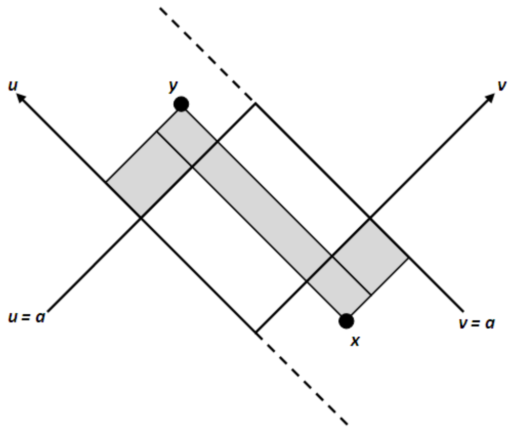


Figure 3.12: volume for integration in Region 1'-3

$$v = v_y.u_y + v_x.u_x - a(u_x + v_y) + a^2$$

$$< n > = \int_0^a dv_y \int_0^a dv_x \int_a^{\infty} du_y \int_{-\infty}^0 du_x \, e^{-u_y.v_y - u_x.v_x + a(u_x + v_y) - a^2}$$

$$< n > = \int_0^a dv_y \int_0^a dv_x \, \frac{1}{a - v_x} \frac{1}{v_y} e^{-a^2}$$

$$< n > = \int_0^a dv_y \frac{e^{-a^2}}{v_y} ln(1 - v_y/a)$$

Using the series expansion for small x in $ln(1 - x)$ again:

$$< n > = \int_0^a dv_y \frac{e^{-a^2}}{v_y} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{v_y}{a} \right)^n$$

$$< n > = e^{-a^2} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{v_y^n}{a^n} \right]_0^a$$

$$< n > = \frac{\pi^2}{6} e^{-a^2}$$

Region 2'-3

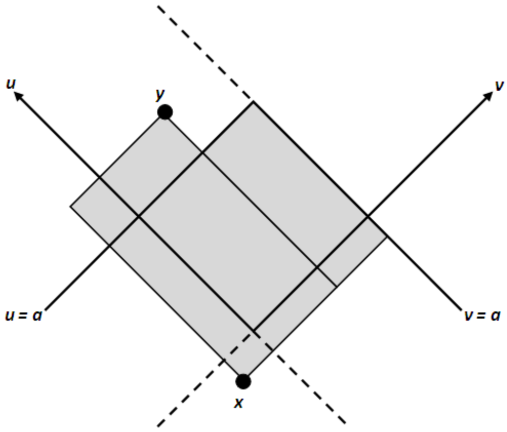


Figure 3.13: volume for integration in Region 2'-3

$$v = v_y.u_y - v_x.u_y + v_x.u_x - a(u_x + v_y) + a^2$$

$$< n >= \int_{-\infty}^{0} dv_x \int_{0}^{a} dv_y \int_{a}^{\infty} du_y \int_{-\infty}^{0} du_x \, e^{-v_y.u_y + v_x.u_y - v_x.u_x + a(u_x + v_y) - a^2}$$

$$< n >= \int_{-\infty}^{0} dv_x \int_{0}^{a} dv_y \, \frac{1}{a - v_x} \frac{1}{v_y - v_x} e^{-a(a - v_x)}$$

$$< n >= \int_{-\infty}^{0} dv_x \, \frac{1}{a - v_x} \, ln\left(\frac{a - v_x}{-v_x}\right) e^{-a(a - v_x)}$$

$x = a - v_x$

$$< n >= -\int_{-\infty}^{a} dx \, \frac{1}{x} \, ln\left(\frac{x}{x - a}\right) e^{-a(x)}$$

$$< n >< e^{-a^2} \int_{-\infty}^{a} \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{a}{x}\right)^n < \frac{\pi^2}{6} e^{-a^2}$$

Region 2'-2

The volume for this is the same as for the causal diamond and the min y case. Therefore the contribution will be the same as in these earlier cases and is suppressed as shown earlier.

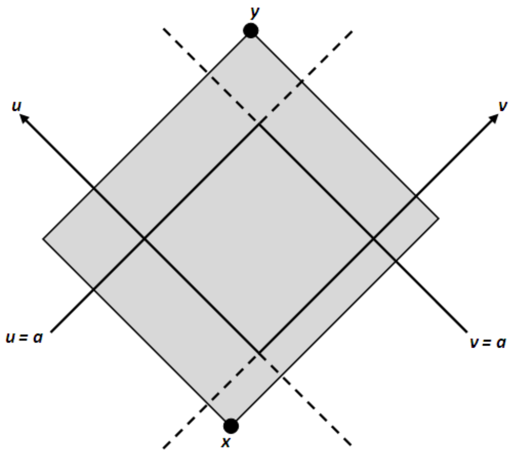


Figure 3.14: volume for integration in Region 2'-2

3.4 Problems in 4D

It appears so far that a although a straight forward programme of simply counting the number of links that cross the horizon and the hypersurface fails due to a divergent answer, there does exist links satisfying two conditions: a minimality condition on y in $J^+(\Sigma)$ and a maximality condition on x in $J^-(H \cap \Sigma)$ that when counted for points $x \in J^-(H \cap \Sigma)$ and $y \in J^+(H \cap \Sigma)$ gives a finite answer in the dimensionally reduced case. Furthermore it appears that this answer is the same for two different types of Black Holes, Shwarzchild and the collapsing null shell. This gives strong evidence for the initial claim.

But the counting procedure fails when higher dimensions are considered. This is due to the fact that the intersection of the future light cone of any point $x \in J^-(H \cap \Sigma)$ and also whose time coordinate is greater than 0 will have an infinite intersection with the horizon. Along this intersection y can be close to both the future light cone of x and the horizon. In this situation the volume will be small and due to the infinite number of points along this intersection will give rise to a divergent answer. There is the possibility that this divergence may counteract the suppression shown in the case of y min and x max but there is no evidence and little hope due to the suppression being seemingly unrelated to the divergence seen here.

Therefore in higher dimensions our counting method requires another addition that suppresses contributions close to $J^+(x) \cap H$. One of these methods the 4-diamond is examined below.

4 The 4-Diamond

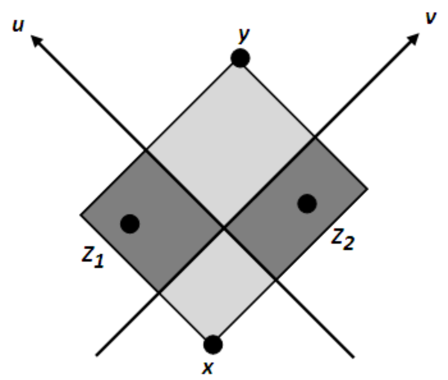


Figure 4.1: Volumes for $z_1$and z_2 are shown in a darker shading

To suppress the divergence in higher dimensions the 4-diamond is defined as follows. It has the same conditions as in the min y case but now 2 extra spacetime points are added. Defined as:

$$z_1 \in \Big(J^+(H) \cap J^+(\Sigma) \cap J^+(x) \cap J^-(y)\Big)$$

$$z_2 \in \Big(J^-(H) \cap J^-(\Sigma) \cap J^+(x) \cap J^-(y)\Big)$$

Now the points x and y no longer form a link. The sum needed to define this new condition will be defined by requiring the old volume minus the volume these points exist in to be empty as before. But now the volume for z_1 and z_2 must contain one and only one extra point each. Following the definition of the poison distribution given earlier the probability of finding only one point in these areas is given by:

$$\rho = V(z_1).V(z_2)\, e^{-V(z_1)-V(z_2)}$$

The number of links will now therefore be calculated by multiplying the probability of finding one point in the above volumes with the probability of finding no points in the remaining volume. Therefore the number of links will be calculated using an integral of the form:

$$< n > = \int_D dV_x dV_y \; V(z_1).V(z_2)\, e^{-V}$$

The addition of these extra points suppress the divergence seen in higher dimensions because when the point y is close to the future light cone of x (or x is close to the past light cone of y) the volume which z_1 must be contained in goes to zero. This eliminates the contribution from these cases and thus suppresses the divergence. The point z_1 is enough to eliminate this divergence but the point z_2 is included for symmetry.

Table 4.1: Volume for z_1 and z_2

Region	Vol for z_1	Vol for z_2
1'-1	$(a - v_x)u_y$	$(a - v_y)u_x$
1'-2	$(a - v_x)a$	$(a - v_y)u_x$
1'-3	$(v_y - v_x)a$	0
2'-1	$a.u_y$	$(a - v_y)u_x$
2'-2	a^2	$(a - v_y)u_x$
2'-3	$a.v_y$	0

The volume for z_1 and z_2 depends on which region the points x and y are in. These volumes are shown in Table 4.1. the volume for z_2 is 0 when $v_y < a$. This means that the inclusion of z_2 will eliminate these contributions completely.

Because most of the links we count are close to H or Σ the addition of either z_1 or z_2 should only act to further reduce any previous contribution. This is because close to H or Σ the volume these points can exist in reduces to zero thus reducing these contributions. This implies that contributions already shown to be suppressed should be further suppressed with the addition of these extra points.

4.1 4-Diamond: min y

Region 1'-1

The volumes for z_1 and z_2 in this region are given in Table 4.1. The volume in the exponential is the same as in the original min y calculation. So the integral is:

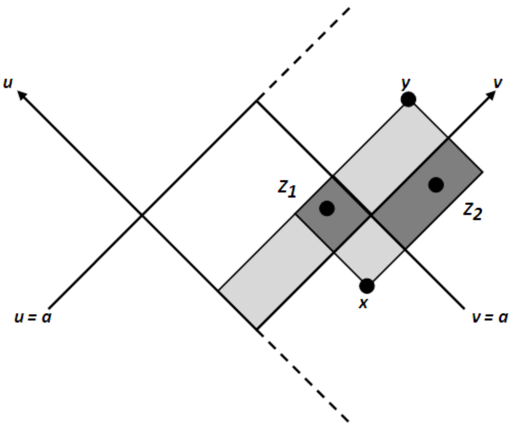


Figure 4.2: Volume for 4-Diamond with min y in region 1'-1

$$< n >= \int_{-\infty}^{0} du_x \int_{0}^{a} dv_x \int_{0}^{a} du_y \int_{a}^{\infty} dv_y \, (a - v_x) u_y (a - v_y) u_x e^{u_x (v_y - v_x) - u_y . v_y}$$

$$< n >= \int_{0}^{a} dv_x \int_{a}^{\infty} dv_y \frac{1}{v_y^2 (v_y - v_x)^2} (a - v_x)(v_y - a) \Big(1 - (a.v_y + 1) e^{-a.vy} \Big)$$

$$I_a := \int_{0}^{a} dv_x \int_{a}^{\infty} dv_y \frac{(a - v_x)(v_y - a)}{v_y^2 (v_y - v_x)^2}$$

$$I_a = \int_{a}^{\infty} dv_y \frac{(v_y - a)}{v_y^2} \int_{0}^{a} dv_x \Big(\frac{(v_y - v_x)}{(v_y - v_x)^2} + \frac{(a - v_y)}{(v_y - v_x)^2} \Big)$$

$$I_a = \int_{a}^{\infty} dv_y \frac{(v_y - a)}{v_y^2} \Big(ln \Big(\frac{v_y}{v_y - a} \Big) - \frac{a}{v_y} \Big)$$

$$I_1 := \int_{a}^{\infty} dv_y \, \frac{a}{v_y^3} (v_y - a) = - \frac{1}{2}$$

$$I_2 := \int_{a}^{\infty} dv_y \frac{1}{v_y^2} (v_y - a) ln \Big(\frac{v_y}{v_y - a} \Big)$$

$$I_2 = \int_{a}^{\infty} dv_y \frac{1}{v_y^2} (v_y - a) \sum_{n=1}^{\infty} \frac{1}{n} \Big(\frac{a}{v_y} \Big)^n$$

$$I_2 = \sum_{n=1}^{\infty} \Big(\frac{1}{n^2} - \frac{1}{n(n+1)} \Big)$$

$$I_2 = \frac{\pi^2}{6} - 1$$

Therefore

$$I_a = I_1 + I_2 = \frac{\pi^2}{6} - \frac{3}{2}$$

$$I_b := \int_0^a dv_x \int_a^\infty dv_y \frac{(a - v_x)(v_y - a)}{v_y^2(v_y - v_x)^2}(av_y + 1)e^{-a.v_y}$$

$$I_b \leq \frac{1}{a}\int_a^\infty dv_y \, (av_y + 1)e^{-a.v_y}$$

$$I_b \leq \frac{-1}{a^2}\Big[(a.v_y + 2)e^{-a.v_y}\Big]_a^\infty$$

$$I_b \leq \frac{1}{a^2}(a^2 + 2)e^{-a^2}$$

This is suppressed. Therefore in the limit of large a:

$$< n >= I_a + I_b \to \frac{\pi^2}{6} - \frac{3}{2}$$

4.2 On the maximality condition on x

Also of interest here is the effect of the addition of z_1 and z_2 onto the earlier condition of just min y without adding any maximality condition on x. Without z_1 and z_2 it was shown earlier that without the condition on x infinities were found in regions 1'-3 and 2'-1.

With z_2 the infinity in region 1'-3 is immediately eliminated since if $v_y < a$ then the volume for z_2 as currently defined is non existent. Therefore contributions from this region no longer exist.

The divergence in region 2'-2 was due to the ability of u_x and u_y to be close to 0 producing a small volume between them as v_y and v_x ran to infinity and -infinity respectively. But with the addition of z_1 u_y can no longer approach 0 and therefore the volume between the points can no longer become small and the divergence may be cured. This is similar to the argument in favour of z_1 curing the divergence in higher dimensions. z_2 should have a similar effect.

Region 2'-1

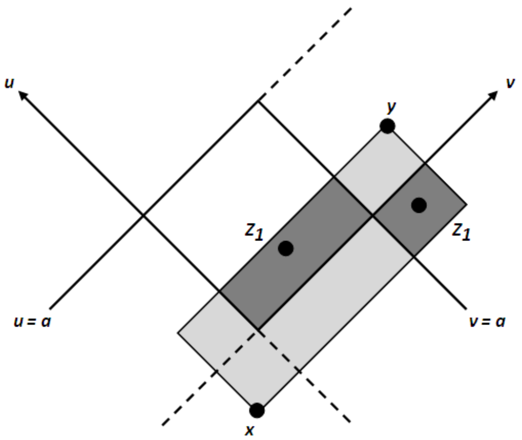


Figure 4.3: Volume for 4-Diamond with min y in region 2'-1

In this region the volumes for z_1 and z_2 are:

$$V(z_1) = a.u_y$$
$$V(z_2) = -u_x(v_y - a)$$

The volume in the exponential is as before the same as for the causal diamond:

$$v = (u_y - u_x).(v_y - v_x)$$

Just considering z_1

$$< n >= \int_{-\infty}^{0} du_x \int_{0}^{a} du_y \int_{a}^{\infty} dv_y \int_{-\infty}^{0} dv_x \, au_y \, e^{u_x(v_y - v_x) - u_y(v_y - v_x)}$$

$$< n >= \int_{0}^{a} du_y \, a.u_y \int_{-\infty}^{0} du_x \, \frac{1}{(u_y - u_x)^2} e^{-a(u_y - u_x)}$$

$$< n >= \int_{0}^{a} du_y \, a.u_y \int_{u_y}^{\infty} dx \, \frac{1}{x^2} e^{-a.x}$$

$$< n >= \int_{0}^{a} du_y \, a.u_y \left[a.Ei(-a(x)) + \frac{e^{-a(x)}}{x} \right]_{u_y}^{\infty}$$

$$< n >= \int_{0}^{a} du_y \, a.u_y \left(a.Ei(-a.u_y) + \frac{e^{-a.u_y}}{u_y} \right) = I_1 + I_2$$

$$I_1 := \int_0^a du_y\, a^2 u_y\, Ei(-a.u_y)$$

$$I_1 = a^2\left[(\frac{1}{2a^2} + \frac{a^2}{2})e^{-a.u_y} + \frac{a^4}{2} Ei(-a.u_y) \right]_0^a$$

$$I_1 = \frac{1}{2}(1+a^2)e^{-a^2} + \frac{a^4}{2}Ei(-a^2) - \frac{1}{2}$$

$$I_2 := \int_0^a du_y\, a\, e^{-a.u_y} = 1 - e^{-a^2}$$

Therefore in the limit of large a:

$$< n >= I_1 + I_2 \to \frac{1}{2}$$

Just considering z_2

$$< n >= \int_{-\infty}^0 du_x \int_0^a du_y \int_a^\infty dv_y \int_{-\infty}^0 dv_x\, (-u_x)(v_y - a)e^{u_x(v_y - v_x) - u_y(v_y - v_x)}$$

$$< n >= \int_0^a du_y \int_{-\infty}^0 du_x\, (-u_x)\frac{1}{(u_y - u_x)^3} e^{-a(u_y - u_x)}$$

$$< n >= \int_0^a du_y \int_{u_y}^\infty dx\, (x - u_y)\frac{1}{x^3} e^{-a.x}$$

$$I_1 := \int_0^a du_y \int_{u_y}^\infty dx\, \frac{1}{x^2} e^{-a.x}$$

gives the same contribution as just considering z_1:

$$I_1 = \frac{1}{2} - e^{-a^2} + \frac{1}{2}(1+a^2)e^{-a^2} + \frac{a^4}{2}Ei(-a^2)$$

$$I_2 := \int_0^a du_y \int_{u_y}^\infty dx\, (-u_y)\frac{1}{x^3} e^{-a.x}$$

$$I_2 = \int_0^a du_y \frac{u_y}{2}\left[a^2 Ei(-a.x) + e^{-a.x}(\frac{a}{x} - \frac{1}{x^2}) \right]_\infty^{u_y}$$

$$I_2 = \int_0^a du_y \frac{1}{2}\Big(a^2 u_y Ei(-a.u_y) + e^{-a.u_y}(a - 1/u_y) \Big)$$

$$I_a := \int_0^a du_y\, a^2 u_y\, Ei(-a.u_y)$$

$$I_a = a^2\left[(\frac{1}{2a^2} + \frac{a^2}{2})e^{-a.u_y} + \frac{a^4}{2}Ei(-a.u_y)\right]_0^a$$

$$I_a = \frac{1}{2}(1+a^2)e^{-a^2} + \frac{a^4}{2}Ei(-a^2) - \frac{1}{2}$$

$$I_b := \int_0^a du_y\, a\, e^{-a.u_y} = 1 - e^{-a^2}$$

$$I_c := \int_0^a du_y\, \frac{1}{u_y} e^{-a.u_y}$$

$$I_c = \Big[Ei(-a.u_y)\Big]_0^a = Ei(-a^2) + \infty$$

Therefore:

$$< n >= I_1 + I_a + I_b + I_c = \infty$$

The infinity is still there because the volume for z_2 is unbounded. In principle z_2 should suppress the divergence by preventing u_x from approaching 0. But it fails to do this. This is because as u_x approaches 0 $(v_y - a)$ can grow to counteract it. This means that the volume for z_2 can still be finite as u_x approaches 0.

This does not occur with z_1 because it exists in a bounded region. The v dimension of the area is bounded by the line $v = 0$ so the maximum length will be a as used above. So as u_y approaches 0 the v component can not grow to counteract this and the volume for z_1 goes to zero and suppresses these contributions.

A similar scenario will occur when attempting to suppress the divergence in higher dimensions. In higher dimensions z_1 is still bounded so should be sufficient. But z_2 is again unbounded and will not be able to suppress the divergence. This means that z_2 is there for purely cosmetic reasons if a maximality condition on x is used as well.

The whole 4-Diamond

$$< n >= \int_{-\infty}^0 du_x \int_0^a du_y \int_a^\infty dv_y \int_{-\infty}^0 dv_x\, a.u_y(-u_x)(v_y - a)e^{u_x(v_y-v_x)-u_y(v_y-v_x)}$$

$$< n > = \int_{-\infty}^{0} du_x \int_{0}^{a} du_y \int_{a}^{\infty} dv_y \, a.u_y(-u_x)(v_y - a)\frac{1}{(u_y - u_x)} e^{u_x.v_y - u_y.v_y}$$

$$< n > = \int_{-\infty}^{0} du_x \int_{0}^{a} du_y \, a.u_y(-u_x)\frac{1}{(u_y - u_x)^3} e^{-a(u_y - u_x)}$$

$$< n > = \int_{0}^{a} du_y \, a.u_y \int_{\infty}^{u_y} dx \, \frac{(u_y - x)}{x^3} e^{-a.x}$$

$$< n > = \int_{0}^{a} du_y \, a.u_y \left[a.Ei(-a.x) + \frac{e^{-a.x}}{x} + \frac{u_y}{2}\Big(a^2 Ei(-a.x) + \frac{e^{-a.x}}{x}(a - 1/x)\Big) \right]_{\infty}^{u_y}$$

$$< n > = \int_{0}^{a} du_y \, a.u_y \left[a.Ei(-a.u_y) + \frac{u_y}{2} a^2 Ei(-a.u_y) + \frac{e^{-a.y}}{2}(a + 1/u_y) \right]$$

$$< n > = \frac{1}{6} + \frac{1}{6}\Big(2a^2 + a^4 - 1\Big)e^{-a^2} - \frac{1}{6}a^6 Ei(a^2) - \frac{1}{2}a^4 Ei(a^2)$$

In the limit of large a this goes to 1/6.

This as expected from the calculation involving just Z_1 gives a finite result. This raises the question of wether a maximality condition on x is required at all. Using both z_1 and z_2 together appear to eliminate the divergences that a maximality condition on x was introduced to avoid. But the philosophy behind link counting is the attribution of black hole entropy to horizon molecules. It therefore makes sense that these horizon molecules are close to the horizon. The horizon stops at the origin so the links coming from region 2'-2 are generally far away from the horizon. On these grounds these links should be discarded and will be if a maximality condition on x is used.

On a more basic level of merely constructing a link counting procedure that gives a finite answer for different types of black holes the fact that this contribution is finite means that maybe it should not be discarded. If z_1 and z_2 are included the maximality condition on x is an added condition which is unnecessary and may add an added complication to the calculation. This is the case of a spacelike hypersurface which will be examined below.

5 Flat Spacetime with Spacelike Hypersurface

Under a suitable lorentz boost the null (or at least the close to null) hypersurface used earlier to measure the entropy will become spacelike. Under this lorentz boost the area defined by the intersection of the horizon and the hypersurface will remain constant. Therefore so too does the value of black hole entropy for a given hypersurface. This means that the application of our procedure to a spacelike hypersurface should give the same answer.

5.1 4-diamond

The volume for z_1 and z_2 now depends on whether the points $A : (u_y, v_x)$ and $B : (u_x, v_y)$ are above or below the hypersurface. the volumes for all regions and the diagrams for region 1'-1 are shown below.

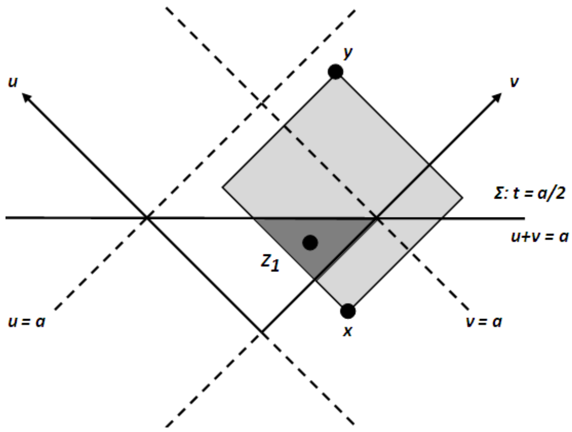
Table 5.1: Volume for z_1

Region of x and y	$A \geq \Sigma$:	$A \leq \Sigma$
1'-1	$\frac{1}{2}(a - v_x)^2$	$u_y(a - v_x) - \frac{1}{2}u_y^2$
1'-2	$\frac{1}{2}(a - v_x)^2$	$a(a - v_x) - \frac{1}{2}a^2$
2'-1	$\frac{1}{2}a^2$	$a.u_y - \frac{1}{2}u_y^2$
2'-2	$\frac{1}{2}a^2$	$\frac{1}{2}a^2$
$v_x \geq a$	0	0

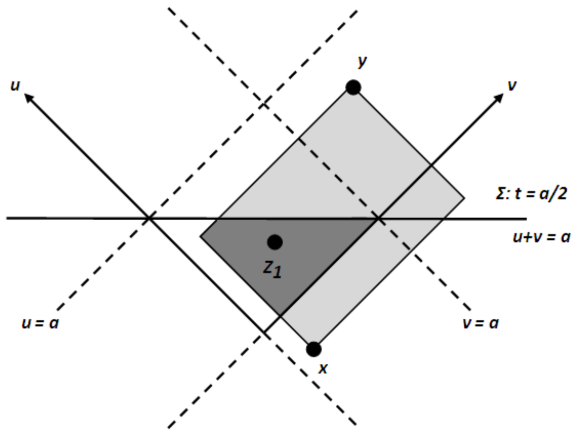
Table 5.2: Volume for z_2

Region of y	$B \geq \Sigma$:	$B \leq \Sigma$
$v_y \geq a$	$-u_x(v_y - a) - \frac{1}{2}u_y^2$	$\frac{1}{2}(v_y - a)^2$
$v_y \leq a$	0	0

This dependance on the points A and B splits each contribution into 4 parts. The inclusion of the points z_1 and z_2 means there is no contribution from $v_y \leq a$

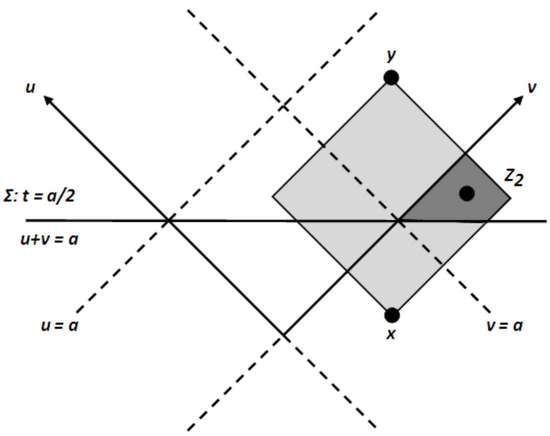


(a) $A \geq \Sigma$

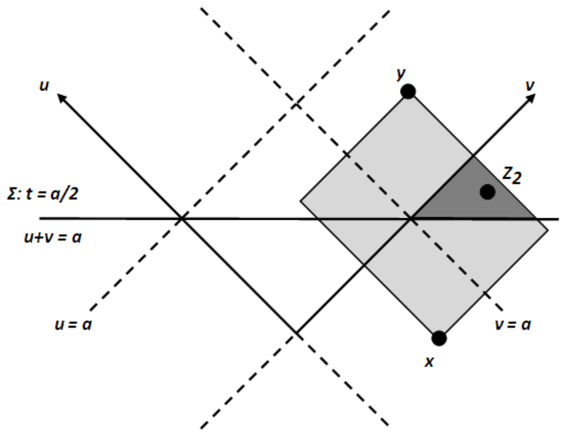


(b) $A \leq \Sigma$

Figure 5.1: Volumes for z_1 in Region 1'-1



(a) $B \geq \Sigma$



(b) $B \leq \Sigma$

Figure 5.2: Volumes for z_2 in Region 1'-1

and $v_x \geq a$. This forces x and y to be either bounded by the line $v = a$, which earlier defined the hypersurface for the null case. This reduces the integration regions to areas similar to the null case. The extra freedom allowed in the movement of x and y due to the hrpersurface being null is counteracted by z_1 and z_2. Although the volumes in which z_1 and z_2 can exist in are different to the null case. Also if the condition that x is maximal in $J^-(\Sigma) \cap J^-(H)$ is included the volume corresponding to this condition will be different to the null case.

Region 1'-1

$A \leq \Sigma$, $B \geq \Sigma$

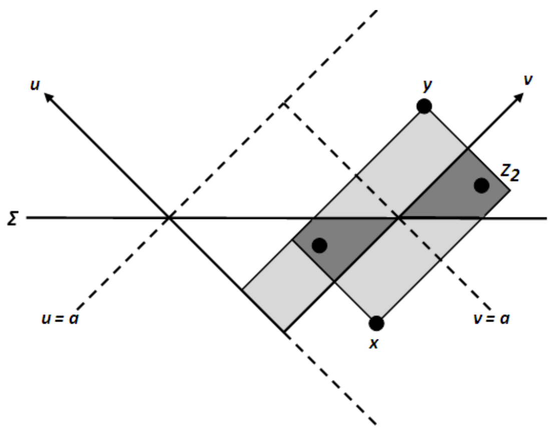


Figure 5.3: $A > \Sigma$, $B < \Sigma$

These conditions on the points A and B translate into the following constraints:

$$A: \; v_x + u_y \leq a$$

$$B: \; v_y + u_y \geq a$$

The condition that x is maximal in $J^-(\Sigma) \cap J^-(H)$ is redundant here as the extra volume is already included in the causal diamond.

The volume in the exponential is the same as in the null case. The volume for z_1 is the given in Table 5.1 and the volume for z_2 is given in Table 5.2. This gives the integral for this contribution as:

$$< n > = -\int_{-\infty}^{0} du_x \int_0^a du_y \int_a^{\infty} dv_y \int_0^a dv_x \left(-\frac{1}{2}u_y^2 + u_y(a - v_x)\right)\left(\frac{1}{2}u_x^2 + u_x(v_y - a)\right) e^{u_x(v_y - v_x) - u_y . v_y}$$

This integral is hard to solve but can be shown to converge to $\frac{\pi^2}{6} - \frac{3}{2}$ for large a numerically.

$A \leq \Sigma$, $B \leq \Sigma$

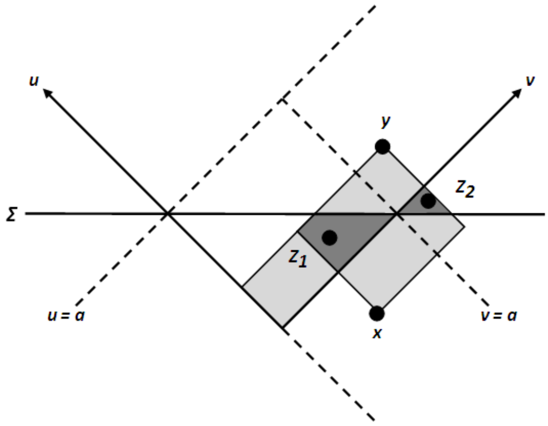


Figure 5.4: $A < \Sigma$, $B < \Sigma$

This contribution has the conditions:

$$A: \; v_x + u_y \leq a$$

$$B: \; v_y + u_y \leq a$$

Since $B \leq \Sigma$ there is a difference between including the maximality condition on x. But the addition of this extra volume in the exponential will only suppress the contribution further. This means that if the contribution is suppressed without the addition of this volume it will still be suppressed with its inclusion. Therefore I will present the calculation without a maximality condition on x first. The volumes for z_1 and z_2 are given in tables 5.1 and 5.2 respectively.

$$<n>= \int_{-\infty}^{a-v_y} du_x \int_0^{a-v_x} du_y \int_a^{\infty} dv_y \int_0^a dv_x \frac{1}{2}(v_y - a)^2 \Big(-\frac{1}{2}u_y^2 + u_y(a-v_x)\Big) e^{u_x(v_y-v_x)-u_y.v_y}$$

$$<n>= \int_0^{a-v_x} du_y \int_a^{\infty} dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y-a)^2}{(v_y-v_y)} \Big(-\frac{1}{2}u_y^2 + u_y(a-v_x)\Big) e^{-(v_y-a)(v_y-v_x)-u_y.v_y}$$

$$<n>= \int_a^{\infty} dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y-a)^2}{(v_y-v_x)} e^{-(v_y-a)(v_y-v_x)} I_{u_y}$$

where:

$$I_{u_y} = \int_0^{a-v_x} du_y \Big(-\frac{1}{2}u_y^2 + u_y(a - v_x)\Big) e^{-u_y.v_y}$$

$$\begin{aligned} I_{u_y} &= -(a - v_x)\frac{1}{v_y^2}\Big[(e^{-v_y(a-v_x)} - 1) + (a - v_x).v_y e^{-v_y(a-v_x)}\Big] \\ &\quad + \frac{1}{2v_y^3}\Big[2(e^{-v_y(a-v_x)} - 1) + 2(a - v_x)v_y e^{-v_y(a-v_x)} + (a - v_x)^2 v_y e^{-v_y(a-v_x)}\Big] \end{aligned}$$

Therefore:

$$\begin{aligned} < n > &= \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y - a)^2}{(v_y - v_x)} e^{-(v_y-a)(v_y-v_x)} \\ &\quad \times \Big[-(a - v_x)\frac{1}{v_y^2}\Big[(e^{-v_y(a-v_x)} - 1) + (a - v_x).v_y e^{-v_y(a-v_x)}\Big] \\ &\qquad + \frac{1}{2v_y^3}\Big[2(e^{-v_y(a-v_x)} - 1) + 2(a - v_x)v_y e^{-v_y(a-v_x)} + (a - v_x)^2 v_y e^{-v_y(a-v_x)}\Big]\Big] \end{aligned}$$

$0 \le v_x \le a$ implies $0 \le (a - v_x) \le a$ therefore:

$$\begin{aligned} < n > &\le \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y - a)^2}{(v_y - v_x)} e^{-(v_y-a)(v_y-v_x)} \\ &\quad \times \Big[-a\frac{1}{v_y^2}\Big[(e^{-v_y(a-v_x)} - 1) + a.v_y e^{-v_y(a-v_x)}\Big] \\ &\qquad + \frac{1}{2v_y^3}\Big[2(e^{-v_y(a-v_x)} - 1) + 2a.v_y e^{-v_y(a-v_x)} + a^2 v_y e^{-v_y(a-v_x)}\Big]\Big] \end{aligned}$$

$$< n > \le I_a + I_b$$

where

$$I_a := \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y - a)^2}{v_y - v_x} e^{-(v_y-a)(v_y-v_x)} \Big(\frac{a}{v_y^2} - \frac{1}{v_y^3}\Big)$$

$$I_b := \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y - a)^2}{v_y - v_x} e^{-v_y^2 + v_x(2v_y - a)} \frac{1}{v_y^3}$$

First I_a:

$$I_a = \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y - a)^2}{v_y - v_x} e^{-(v_y-a)(v_y-v_x)} \Big(\frac{a}{v_y^2} - \frac{1}{v_y^3}\Big)$$

$$I_a \le \int_a^\infty dv_y \, \frac{1}{2}\Big(\frac{a}{v_y^2} - \frac{1}{v_y^3}\Big) e^{-v_y(v_y-a)} \Big[e^{v_x(v_y-a)}\Big]_0^a$$

changing variables with $y = v_y - a$:

$$I_a \le \int_0^\infty dy \, \frac{1}{2}\Big(\frac{a}{a^2} - \frac{1}{a^3}\Big)\Big(e^{-y^2} - e^{-a.y}\Big)$$

since:

$$\int_0^\infty dy \, e^{-y^2} = \frac{\sqrt{\pi}}{2}$$

$$I_a \le \frac{1}{2}\Big(\frac{1}{a} - \frac{1}{a^3}\Big)(\frac{\sqrt{\pi}}{2} - \frac{1}{a})$$

Now looking at I_b:

$$I_b = \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(v_y-a)^2}{v_y - v_x} e^{-v_y^2 + v_x(2v_y - a)} \frac{1}{v_y^3}$$

$$I_b \le \int_a^\infty dv_y \frac{1}{2}\frac{(v_y - a)}{v_y^3}\frac{e^{-v_y^2}}{(2v_y - a)}\Big(e^{a(2v_y - a)} - 1\Big)$$

splitting this further into two integrals:

$$I_b \le I_{b1} + I_{b2}$$

$$I_{b1} := \int_a^\infty dv_y \frac{1}{2a}(\frac{1}{v_y^2} - \frac{1}{v_y^3})e^{-v_y^2}$$

$$I_{b2} := \int_a^\infty dv_y \frac{1}{2a}\frac{(v_y - a)}{v_y^3} e^{-(v_y-a)^2}$$

$$I_{b1} \le \frac{1}{2a}e^{-a^2}\int_a^\infty dv_y(\frac{1}{v_y^2} - \frac{1}{v_y^3})$$

$$I_{b1} \le \frac{1}{2a}e^{-a^2}(\frac{1}{a} - \frac{2}{a^2})$$

using again the change of variable $y = v_y - a$:

$$I_{b2} \le \int_0^\infty dy \frac{1}{2}\frac{y}{a^4}e^{-y^2}$$

$$I_{b2} \le \frac{1}{4a^4}$$

Since I_{b1},I_{b2} and I_a are suppressed for large a so too is $< n >$ As discussed earlier. The suppression of this contribution also implies the suppression of a contribution with a maximality condition on x.

$A \geq \Sigma$, $B \leq \Sigma$

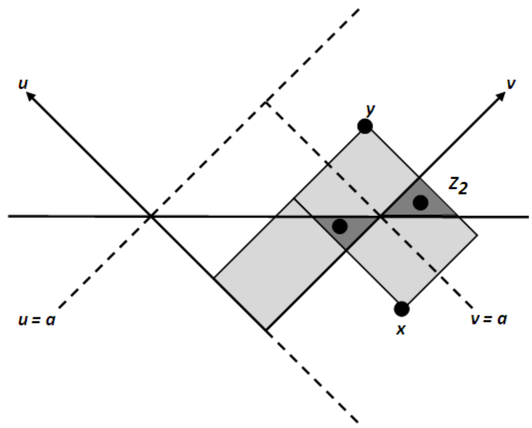


Figure 5.5: $A > \Sigma$, $B < \Sigma$

$$A: \ v_x + u_y \geq a$$

$$B: \ v_y + u_y \leq a$$

Again since $B \leq \Sigma$ the volume in the exponential will depend on whether or not a maximality condition on x is included. As with the earlier case, if the contribution without the extra condition is suppressed then so too will the contribution with the maximality condition.

$$< n >= \int_{-\infty}^{a-v_y} du_x \int_{a-v_x}^{a} du_y \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \, 14(a-v_x)^2 (v_y-a)^2 \, e^{u_x(v_y-v_x)-u_y.v_y}$$

$$< n >= \int_{a-v_x}^{a} du_y \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \, \frac{1}{4} \frac{(a-v_x)^2(v_y-a)^2}{(v_y-v_x)} \, e^{-(v_y-a)(v_y-v_x)-u_y.v_y}$$

$$< n >= \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \, \frac{1}{4} \frac{(a-v_x)^2(v_y-a)^2}{v_y(v_y-v_x)} \, (e^{-v_y^2+v_x(2v_y-a)} - e^{-v_y(v_y-v_x)-a.v_x})$$

$$< n > \leq \int_a^\infty dv_y \int_0^a dv_x \, \frac{1}{4} \frac{a^2(v_y - a)}{v_y} \, (e^{-v_y^2 + v_x(2v_y - a)} - e^{-v_y(v_y - v_x) - a.v_x})$$

$$< n > \leq \int_a^\infty dv_y \, \frac{1}{4} \frac{a^2(v_y - a)}{v_y} \, (e^{-v_y^2} \Big(\frac{e^{a(2v_y - a)}}{(2v_y - a)} - \frac{1}{(2v_y - a)} - \frac{e^{a.(v_y - a)}}{(v_y - a)} + \frac{1}{(v_y - a)} \Big)$$

dealing with this term by term as:

$$< n > \leq I_a + I_b + I_c + I_d$$

$$I_a = \int_a^\infty dv_y \, \frac{1}{4} \frac{a^2(v_y - a)}{v_y} \frac{e^{(v_y - a)^2}}{(2v_y - a)}$$

$$I_a \leq \int_0^\infty dy \, \frac{1}{4} \frac{y}{a} e^{-y^2}$$

$$I_a \leq \frac{1}{8a}$$

$$I_b = \int_a^\infty dv_y \, \frac{1}{4} \frac{a^2(v_y - a)}{v_y} \frac{e^{-v_y^2}}{(2v_y - a)}$$

$$I_b \leq \int_a^\infty dv_y \, \frac{1}{4} (v_y - a) e^{-v_y^2}$$

$$I_b \leq \frac{1}{8} e^{-a^2} - \frac{a}{8} \sqrt{\pi} \, erfc(a)$$

Where $erfc(x)$ is the error function. This is suppressed for large a.

$$I_c = \int_a^\infty dv_y \, \frac{1}{4} \frac{a^2(v_y - a)}{v_y} \frac{e^{-(v_y - 1/2)^2 - 3a^2/2}}{(v_y - a)}$$

$$I_c \leq \int_{a/2}^\infty dy \frac{1}{4} a e^{-y^2} e^{-3a^2/2}$$

$$I_c \leq \frac{1}{4} a \, erfc(a/2) e^{-31^2/2}$$

$$I_d = \int_a^\infty dv_y \, \frac{1}{4} \frac{a^2(v_y - a)}{v_y} \frac{e^{-v_y^2}}{(v_y - a)}$$

$$I_d \leq \int_a^\infty \frac{1}{4} a\, e^{-v_y^2}$$

$$I_d \leq \frac{1}{4} a\, erfc(a)$$

All these terms are suppressed for large a. Therefore $< n >$ is suppressed as well.

$A \geq \Sigma$ **,** $B \geq \Sigma$

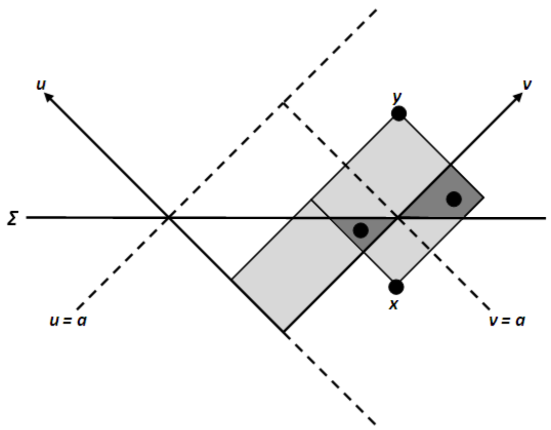


Figure 5.6: $A > \Sigma$, $B > \Sigma$

$$< n > = \int_{a-v_y}^{0} du_x \int_{a-v_x}^{a} du_y \int_a^\infty dv_y \int_0^a dv_x \, -\frac{1}{2}(a-v_x)^2 \Big(\frac{1}{2}u_x^2 + u_x(v_y - a)\Big)\, e^{u_x(v_y - v_x) - u_y . v_y} \tag{5.1}$$

Splitting this up into two integrals:

$$< n > = I_a + I_b$$

$$I_a := \int_{a-v_y}^{0} du_x \int_{a-v_x}^{a} du_y \int_a^\infty dv_y \int_0^a dv_x \, -\frac{1}{2} u_x (a-v_x)^2 (v_y - a)\, e^{u_x(v_y - v_x) - u_y . v_y}$$

$$I_b := \int_{a-v_y}^{0} du_x \int_{a-v_x}^{a} du_y \int_a^\infty dv_y \int_0^a dv_x \, -\frac{1}{4} u_x^2 (a - v_x)^2\, e^{u_x(v_y - v_x) - u_y . v_y}$$

Looking at I_a first:

$$I_a = \int_{a-v_x}^{a} du_y \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \, \frac{1}{2}(a-v_x)^2(v_y-a) \, \frac{e^{-u_y.v_y}}{(v_y-v_x)} \Big[(\frac{1}{(v_y-v_x)} - u_x))e^{u_x(v_y-v_x)} \Big]_{a-v_y}^{0}$$

$$I_a = \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \, \frac{1}{2} \frac{(a-v_x)^2(v_y-a)}{v_y(v_y-v_x)^2} \, (e^{-v_y(a-v_x)} - e^{-a.v_y}) \\ \times \Big[1 - \Big(1 + (v_y-v_x)(v_y-a) \Big) e^{(v_y-v_x)(a-v_y)} \Big]$$

$-v_y(a-v_x) \geq -a.v_y$ therefore $e^{-v_y(a-v_x)} \geq e^{-a.v_y}$. This means that if terms due to the first exponential are suppressed so will terms due to the second exponential. Therefore it is enough to consider:

$$I_1 := \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \, \frac{1}{2} \frac{(a-v_x)^2(v_y-a)}{v_y(v_y-v_x)^2} \, e^{-v_y(a-v_x)} \\ \times \Big[1 - \Big(1 + (v_y-v_x)(v_y-a) \Big) e^{(v_y-v_x)(a-v_y)} \Big]$$

examining this term by term:

$$I_{11} := \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{1}{2} \frac{(a-v_x)^2(v_y-a)}{v_y(v_y-v_x)^2} e^{-v_y(a-v_x)}$$

$$I_{11} \leq \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{1}{2} \frac{(a-v_x)}{v_y} e^{-v_y(a-v_x)}$$

changing variables as $x = a - v_x$ and integrating over x gives;

$$I_{11} \leq \int_{a}^{\infty} dv_y \frac{1}{2v_y^3} \Big(1 - (1 + a.v_y)e^{-a.v_y} \Big)$$

using:

$$\int \frac{e^{-a.x}}{x^2} \, dx = a.Ei(a.x) - \frac{e^{-a.x}}{x}$$

and

$$\int dx \frac{e^{-a.x}}{x^2} \, dx = -\frac{1}{2} \Big(a^2.Ei(a.x) + e^{-a.x}(\frac{1}{x^2} - \frac{a}{x}) \Big)$$

this integral can be solved:

$$I_{11} \leq \frac{1}{4}\Big(\frac{1}{a^2} + a^2 Ei(a^2) - (1+\frac{1}{a^2})e^{-a^2}\Big)$$

This is suppressed for large a.

$$I_{12} := \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(a-v_x)^2(v_y-a)}{v_y(v_y-v_x)^2} e^{-(v_y-v_x)^2-v_x(a-v_x)}$$

$$I_{12} \leq \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(a-v_x)^2}{a(v_y-v_x)} e^{-(v_y-v_x)^2-v_x(a-v_x)}$$

Using the substitutions $y = v_y - v_x$ and $x = a - v_x$ gives:

$$I_{12} \leq \int_0^a dx\, \frac{x^2}{2a} e^{-x.v_x} \int_x^\infty dy\, \frac{e^{-y^2}}{y}$$

integrating over y and using $-x.v_x \leq 0$ gives:

$$I_{12} \leq \frac{1}{4a}\int_0^a dx\, x^2 Ei(x^2)$$

$$I_{12} \leq \frac{1}{24a}\Big(a^2 Ei(a^2) + a\sqrt{\pi}\, erf(a) - 2e^{-a^2}\Big)$$

This is suppressed for large a.

$$I_{13} := \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2}\frac{(a-v_x)^2(v_y-a)^2}{v_y(v_y-v_x)} e^{-(v_y-v_x)^2-v_x(a-v_x)}$$

$$I_{13} \leq \int_a^\infty dv_y \frac{a}{2a}(v_y-a)e^{-v_y^2} \int_0^a dv_x (a-v_x)e^{v_x(2v_y-a)}$$

$$I_{13} \leq \int_a^\infty dv_y\, \frac{1}{2}\frac{(v_y-a)}{(2v_y-a)^2}\Big(e^{-(v_y-v_x)^2} + \Big(1-a(2v_y-a)\Big)e^{-v_y^2}\Big)$$

$$I_{13} \leq \int_a^\infty dv_y\, \frac{1}{2}\frac{(v_y-a)}{a^2}\Big(e^{-(v_y-v_x)^2} + \Big(1-a(2v_y-a)\Big)e^{-v_y^2}\Big)$$

$$I_{13} \leq \frac{1}{4}\Big(\frac{1}{a^2} + a\sqrt{\pi}\, erf(a) - a\sqrt{\pi} + (1-\frac{1}{a^2})e^{-a^2}\Big)$$

The error function tends to one for large a so this term is suppressed for large a. Since $I_1 = I_{11} + I_{12} + I_{13}$ the suppression of each individual term for large a implies the suppression of I_1 and thus I_a.

Now to show the suppression of I_b:

$$I_b = \int_{a-v_y}^{0} du_x \int_{a-v_x}^{a} du_y \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \ -\frac{1}{4} u_x^2 (a-v_x)^2 \, e^{u_x(v_y-v_x)-u_y.v_y}$$

$$I_b = \int_{a-v_y}^{0} du_x \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \ -\frac{1}{4} u_x^2 \frac{(a-v_x)^2}{v_y} \, e^{u_x(v_y-v_x)}(e^{-v_y(a-v_x)} - e^{-a.v_y})$$

$$\begin{aligned} I_b = \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{(a-v_x)^2}{4v_y(v_y-v_x)^3} \, (e^{-v_y(a-v_x)} - e^{-a.v_y}) \\ \times \Big[-2 + \Big(2 - 2(a-v_y)(v_y-v_x) + (a-v_y)^2(v_y-v_x)^2\Big) e^{(v_y-v_x)(a-v_y)} \Big] \end{aligned}$$

Again since $-v_y(a-v_x) \geq -a.v_y$ implies $e^{-v_y(a-v_x)} \geq e^{-a.v_y}$. This means that if terms due to the first exponential are suppressed so will terms due to the second exponential. Therefore it will be enough to consider:

$$\begin{aligned} I_2 = \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{(a-v_x)^2}{4v_y(v_y-v_x)^3} \, e^{-v_y(a-v_x)} \\ \times \Big[-2 + \Big(2 - 2(a-v_y)(v_y-v_x) + (a-v_y)^2(v_y-v_x)^2\Big) e^{(v_y-v_x)(a-v_y)} \Big] \end{aligned}$$

Again treating it term by term:

$$I_{21} := \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{(a-v_x)^2}{2v_y(v_y-v_x)^3} \, e^{-v_y(a-v_x)}$$

$$I_{21} \leq \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{(a-v_x)^2}{2a(v_y-v_x)^3} \, e^{-a(a-v_x)}$$

$$I_{21} \leq \int_{0}^{a} dv_x \frac{(a-v_x)^2}{4a(a-v_x)^2} \, e^{-a(a-v_x)}$$

$$I_{21} \leq \frac{1}{4a^2}(1-e^{-a^2})$$

$$I_{22} := \int_{a}^{\infty} dv_y \int_{0}^{a} dv_x \frac{(a-v_x)^2}{2v_y(v_y-v_x)^3} \, e^{-(v_y-v_x)^2 - v_x(v_y-a)}$$

$$I_{22} \leq \int_a^\infty dv_y \int_0^a dv_x \frac{1}{2a(v_y - v_x)} \, e^{-(v_y-v_x)^2}$$

$$I_{22} \leq \int_0^a dv_x \frac{1}{4a} Ei\Big((a-v_x)^2\Big)$$

$$I_{22} \leq \frac{1}{4}(Ei(a^2) + \frac{1}{a}\sqrt{\pi}\, erf(a))$$

This is suppressed for large a.

$$I_{23} := \int_a^\infty dv_y \int_0^a dv_x \frac{(a-v_x)^2(a-v_y)}{2v_y(v_y-v_x)^2} \, e^{-(v_y-v_x)^2 - v_x(v_y-a)}$$

$$I_{23} \leq \int_a^\infty dv_y \int_0^a dv_x \frac{(a-v_x)^2}{2a(v_y-v_x)} \, e^{-(v_y-v_x)^2}$$

$$I_{23} \leq \int_0^a dv_x \frac{1}{4a}(a-v_x)^2 Ei\Big((a-v_x)^2\Big)$$

$$I_{23} \leq \frac{1}{24}\Big(2a^2 Ei(a^2) + \frac{1}{a}\sqrt{\pi}\, erf(a) - 2\, e^{-a^2}\Big)$$

This is suppressed for large a.

6 Further Ideas and Conclusion

6.1 4D Case

6.2 De-sitter

6.3 conclusion

Bibliography

[1] It is more convinient and faster to use `bibtex` instead of writing your bibliography manually.

[2] You can even use a tool like `jabref` to manage and maintain your database of references.